MATHEMATICAL RECREATIONS

by Ian Stewart

Cementing Relationships

he prestigious journal *Nature* manages to combine high-powered research papers with eclectic science writing. One of its regular columns is Art and Science, whose name pretty much speaks for itself. In the December 11, 1997, issue, art historian Martin Kemp describes the remarkable landscapes of a London artist named Jonathan Callan. Unlike conventional landscape art, Callan's works are sculptures, not paintings. And his landscapes are unlike anything seen on earth or indeed on any known world. They are three-dimensional forms created by pouring cement onto a perforated board.

Kemp, a professor in the University of Oxford's art history department, notes a relationship between Callan's sculptures and recent work in the field of complexity theory. Certain general principles seem to govern Callan's fantastic landscapes—for instance, the highest peaks of cement occur in the regions farthest from the holes. In a letter to the editor in a later issue of *Nature* (January 29, 1998), Adrian Webster, an astronomer at the Royal Observatory in Edinburgh, points out that the curious geometry of Callan's landscapes can be understood using a more classical branch of mathematics, the theory of Voronoi cells. He also explains how Voronoi cells illustrate one of the major recent discoveries of astronomy, the foamlike distribution of matter in the universe. If ever there was an example of the unity of mathematics, art and science, this has to be it.

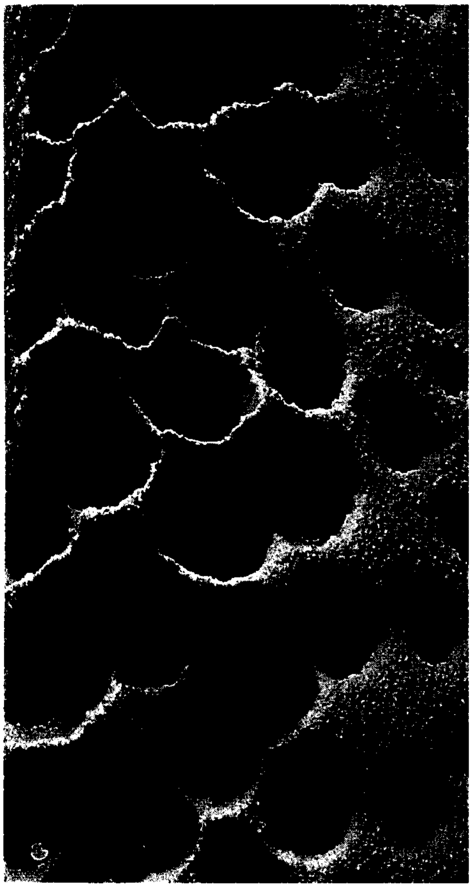
Since the very first cave paintings, artists have relied on processes from physics and chemistry to create their masterpieces. In ancient Greece and Rome, sculptors had to understand how stones fractured and how molten bronze flowed into a cast. Renaissance painters studied the properties of pigments. The traditional artist's technique has been to control these physical processes, using them to shape sculptures and paintings in desired ways. Callan is one of a smaller band of modern artists who relinquish that control. They allow the physical and chemical processes of their media to determine the main features of their artworks.

Callan begins each sculpture by drilling a random pattern of holes into a horizontal board. He then sieves cement powder evenly over the surface. Some cement falls through the holes, and some piles up in the areas between them. The sculpture hardens by absorbing moisture from the air. The final result resembles a moonscape, with jagged peaks surrounding steep craters. Callan compares the sculpture to an earthly mountain range: "A geography that seems both eminently natural and highly artificial—the Alps brand new."

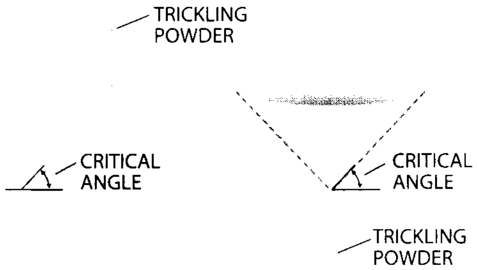
A more apt comparison might liken Callan's artworks to a collection of sandpiles. Civil engineers have long been familiar with how granular materials—such as sand, soil or cement powder—pile up. The simplest and most important feature is the existence of a "critical angle." Depending on the nature of the granular material, there is a steepest slope that it can sustain without collapsing. This slope runs at a constant angle with the ground—the critical angle. If you keep piling sand higher and higher—say, by pouring it in a thin stream—the slope of the sandpile will steepen until it reaches the critical angle. Any extra sand will then trickle down the pile, causing either a tiny avalanche or a big one, to restore the constant slope. The resulting steady-state shape, in this simplest model, is a cone whose sides slope at exactly the critical angle.

Complexity theorists study the process by which the slope attains this shape and the nature of the avalanches, big or small, that accompany its growth. Danish physicist Per Bak coined the term "self-organized criticality" for such processes, and he has suggested that they model many important features of the natural world, especially evolution (where the avalanches involve not grains of sand but entire species, and the piles are in an imaginary space of potential organisms).

In Callan's artworks, the structure of



COURTESY OF HALES GALLERY



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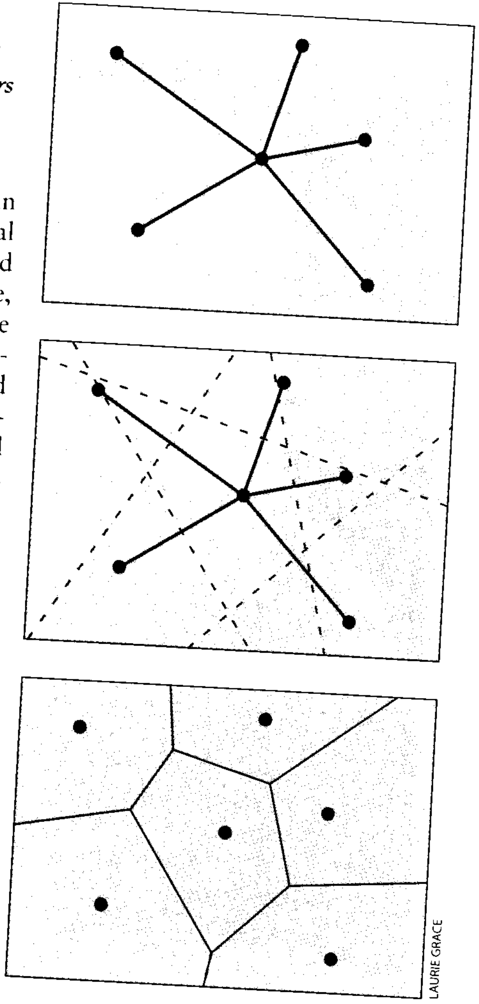
JAGGED CRATERS pock the surface of one of Jonathan Callan's cement sculptures (far left). *The sides of each crater slope at the same angle as the sides of a conical pile of cement powder* (left).

VORONOI CELLS can be sketched by drawing the perpendicular bisectors of the lines between the holes.



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cement powder around each hole is an inversion of the civil engineer's conical sandpile. Consider a horizontal board with just one hole. Away from the hole, cement rises in every direction at the critical angle, creating a conical depression whose tip points downward and rests at the center of the hole [*see illustration on page 100*]. These inverted cones are the craters that form Callan's striking landscapes.

But what of the geometry when there are several holes? The key point now is that any cascading cement powder that trickles through the board will fall out through the hole that is nearest to its initial point of impact. It is therefore possible to predict where the boundaries between the conical craters will occur. Divide the board into regions surrounding the holes, in such a manner that each region consists of those points that are closer to the chosen hole than they are to any other hole. The region is the hole's "sphere of influence," so to speak, except that it is not a sphere but a polygon. Provided the board is horizontal, the boundaries between these regions are directly underneath the common boundaries of adjacent craters.

A way to sketch one of these regions is to choose any hole and draw lines from its center to the centers of all the other holes [*see illustration above*]. Cut each line in half and from that point draw another line at right angles to it (that is, draw the perpendicular bisector). The result will be a crisscrossing network of bisectors. Find the smallest convex region that is bounded by segments of this network and contains the chosen hole. This region is known as a Voronoi cell. Each hole is surrounded by a unique Voronoi cell, and all the cells together tile the plane. Georgii F. Voronoi (1868–1908) was a Russian mathematician who worked on number theory and multidimensional tilings. Voronoi cells go by a few other names—Dirichlet domains and Wigner-Seitz cells, for example—because they have been rediscovered in many contexts.

Callan's craters, then, rise in inverted cones at the same critical angle and meet above the edges of the Voronoi cells defined by his system of holes. One comfortable consequence of this geometry is that when two slopes meet, they do so at the same height above the board—there is no sharp discontinuity. Another feature, less obvious, can also be deduced: the shape of the ridge where one crater merges into its neighbor. In the abstract, what we have are two inverted cones rising at identical angles. They meet above the perpendicular bisector of the line that joins their vertices—above the Voronoi boundary. Consequently, their intersection lies in the vertical plane through that boundary line. What curve do you get if you cut a cone with a vertical plane? The ancient Greeks knew the answer: a hyperbola. This fact helps to explain the rather jagged nature of Callan's landscapes.

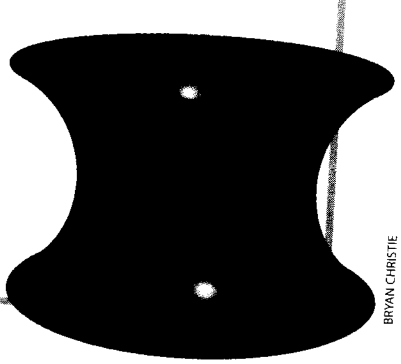
What of the connection with astronomy? Instead of holes in a plane, imagine points in three-dimensional space. In the plane, the perpendicular bisector of a pair of points is a line, but in space it is a plane. Draw these bisecting planes for the lines between a given point and all the other points. Let the Voronoi cell of the given point be the smallest convex region that surrounds it and is bounded by parts of these planes. Now the Voronoi cell is a polyhedron. Astronomers have recently discovered that the large-scale distribution of matter in the universe resembles a network of such polyhedra. Most galactic clusters seem to be located on the boundaries of neighboring Voronoi cells. This pattern has been called the Voronoi foam model of the universe because it looks somewhat like a giant bubble bath.

There is an analogy—imperfect, but still illuminating—with the distribution of cement powder in Callan's landscapes. In his sculptures, the cement piles up highest along the Voronoi boundaries. The analogous property in three-dimensional space would be that as the universe expands, matter concentrates along the same boundaries. So this one simple idea encapsulates some arresting art, elegant mathematics and deep physics about the structure of the universe. SA

FEEDBACK

The column on the Double Bubble Conjecture [January] has given many readers more toil and trouble than they deserve. Several pointed out that one of the bubbles in the accompanying photographs did not have the shape of a minimal-surface catenoid, as stated. A true catenoid of this type (*below*) is stubbier and has a more tightly pinched waist.

Why the difference between the ideal shape and the photograph? In fact, the shape of a bubble that forms between two parallel rings depends on how far apart the rings are, compared with their diameters. If the separation increases beyond a critical value—as happened apparently in the photograph—the bubble elongates and then collapses, forming two disconnected disks, each spanning one ring. —*I.S.*



BRYAN CHRISTIE