

H Appendix: A Useful Normalization Identity

A series whose terms z_m , $m = 1, 2, \dots$ are defined by a recursion relation:

$$z_m = \prod_{i=1}^{m-1} \frac{i}{\alpha + i} \quad (77)$$

can be summed exactly as follows.

Rewrite z_m as:

$$z_m = \frac{\Gamma(m)\Gamma(\alpha + 1)}{\Gamma(\alpha + m)} \quad (78)$$

with the usual definition for the gamma function:

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \quad (79)$$

The integral representation of the beta function $B(x, y)$ provides the key identity to carry out the sum:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 dt t^{x-1} (1 - t)^{y-1} \quad (80)$$

Combining these relations leads to:

$$\sum_{m=1}^{\infty} z_m = \alpha \int_0^1 \sum_{m=1}^{\infty} t^{m-1} (1 - t)^{\alpha-1} \quad (81)$$

$$= \alpha \int_0^1 (1 - t)^{\alpha-2} \quad (82)$$

$$= \frac{\alpha}{\alpha - 1} \quad (83)$$

$$(84)$$