A Appendix: Analytic Solution of the Minimal Model

It helps to introduce a new parameterization of time:

$$u = \log \phi(t) = \log\left(1 + \frac{(R+1)t}{N_0}\right) \tag{20}$$

with associated derivative:

$$\frac{\partial}{\partial t} = \frac{R+1}{N_0 e^u}\frac{\partial}{\partial u} \tag{21}$$

With this definition, $u = 0$ corresponds to $t = 0$.

This new variable helps rid the differential equations (2) of explicit time dependence:

$$\begin{aligned} \frac{\partial F(m,u)}{\partial u} + \frac{mF(m,u)}{R+1} &= \frac{(m-1)F(m-1,u)}{R+1} \quad (m > 1) \\ \frac{\partial F(1,u)}{\partial u} + \frac{F(1,u)}{R+1} &= \frac{N_0 R}{R+1}e^u \end{aligned} \tag{22}$$

Note that the equation the special bin $F(1,u)$ does not depend on any other $F(m,u)$, so it can be solved separately. Once it is known, the solution for any other m can be found by successive integration:

$$F(m+1,u) = \exp\left(-\frac{m+1}{R+1}u\right)\int_0^u \frac{dv}{R+1}\, mF(m,v)\exp\left(\frac{m+1}{R+1}v\right) \quad \text{for } m+1 = 2,3,\cdots \tag{23}$$

The solution for m serves as a "source" for $m+1$. The relation (23) follows by multiplying both sides of (2) by $\exp\left(\frac{m+1}{R+1}u\right)$ and integrating. Note that this solution ensures that $F(m > 1, t = 0) = 0$, so the initial conditions are automatically satisfied. Our method of solving the differential equation is elementary and standard, see (Simmons, 1994) for more details.

The solution for $m = 1$ can be found in the same way:

$$\frac{\partial}{\partial u}\left[\exp\left(\frac{u}{R+1}\right)F(1,u)\right] = \exp\left(\frac{u}{R+1} + u\right)\frac{N_0 R}{R+1} \tag{24}$$

$$F(1,u) = N_0 \exp\left(-\frac{u}{1+r}\right) + \frac{N_0 R}{R+2}\left[\exp u - \exp\left(-\frac{u}{1+r}\right)\right] \tag{25}$$

The full solution follows by successive application of (23). There are two types of integrals that come up:

$$\begin{aligned} &\exp\left(-\frac{m+1}{R+1}u\right)\int_0^u \frac{dv}{R+1}\,[m\exp v]\exp\left(\frac{m+1}{R+1}v\right) \\ &\qquad = \frac{m}{R+2+m}\left[\exp u - \exp\left(-\frac{m+1}{R+1}u\right)\right] \end{aligned} \tag{26}$$

$$\begin{aligned} &\exp\left(-\frac{m+n+1}{R+1}u\right)\int_0^u \frac{dv}{R+1}(m+n) \\ &\qquad \left[\exp\left(-\frac{mv}{R+1}\right)\left(1-\exp\left(-\frac{v}{R+1}\right)\right)^n\right]\exp\left(\frac{m+n+1}{R+1}v\right) \\ &\qquad = \frac{m+n}{n+1}\left[\exp\left(-\frac{mu}{R+1}\right)\left(1-\exp\left(-\frac{u}{R+1}\right)\right)^{n+1}\right] \end{aligned} \tag{27}$$

The coefficients that emerge from these integrations define the recursion relations for A_m and β_n^m:

$$A_{m+1} = \frac{m}{R+2+m} A_m \tag{28}$$

$$\beta_{n+1}^m = \frac{m+n}{n+1} \beta_n^m \tag{29}$$

The full solution to (2), taking into account the initial conditions, is given by:

$$F(m,t) = N_0 \phi^{-\frac{1}{1+R}} \left(1 - \phi^{-\frac{1}{1+R}}\right)^{m-1} + A_m(\phi - \phi^{-\frac{m}{1+r}}) - \sum_{i=1}^{m-1} A_i \beta_{m-i}^i \phi^{-\frac{i}{1+R}} \left(1 - \phi^{-\frac{1}{1+R}}\right)^{m-i}$$

$$A_m = \frac{RN_0}{R+2} \prod_{i=1}^{m-1} \frac{i}{R+2+i} = RN_0 \frac{\Gamma(m)\Gamma(R+2)}{\Gamma(R+2+m)}$$

$$\beta_n^m = \prod_{k=1}^{n} \frac{m+k-1}{k} = \frac{(m+n-1)!}{(m-1)!n!} \tag{30}$$

with the understanding that an empty product is unity, i.e. $\prod_{i=1}^{0} f(i) = 1$.

Note that the coefficients A_m and β_n^m do not depend on time, and furthermore β_n^m has no dependence on R or N_0. The product of coefficients $A_i \beta_{m-i}^i$ can be simplified:

$$A_i \beta_{m-i}^i = \frac{(m-1)!}{(m-i)! \prod_{j=1}^{i-1}(R+2+j)} \tag{31}$$

but it will be useful keep these these coefficients separate when considering the solution for more general initial conditions. Note that we use the standard definition for the gamma function $\Gamma(x)$; see Appendix H.

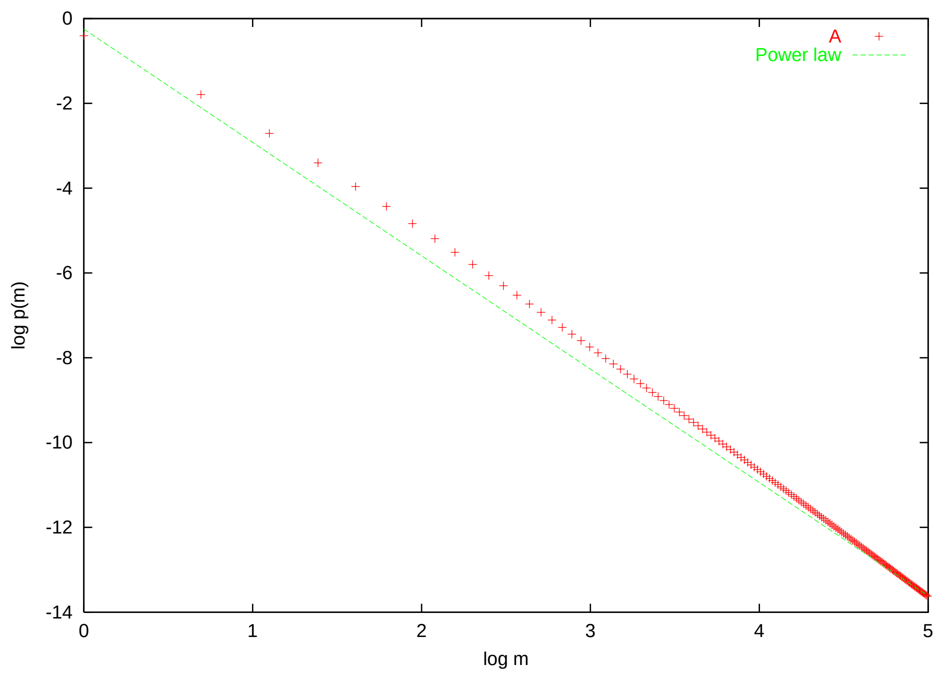

Figure 6: The normalized A_m coefficients (points) and a power-law fit (line), shown as a log-log plot as a function of size m, for $R = 1$.